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RADAR AMBIGUITY ANALYSIS

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RADAR AMBIGUITY ANALYSIS

by P. M. Woodward

Summary

Methods are given for obtaining the ambiguity characteristics of various types of high-resolution radar waveform, amplitude or phase modulated, pulsed or CW. The analysis deals mainly with random modulations, to which many complicated types of waveform approximate.

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G	Gaussian function, section 6 (a) and (b)
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P	Pulse-length, eqn. 42
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W	Signal bandwidth
X	Ambiguity function for amplitude noise, eqn. 52
Y	Ambiguity function for phase noise, eqn. 77, 78
Z	Ambiguity function for complex noise, eqn. 36
a,b	Signal modulation constants, eqns. 6, 23, 24
c	Auto-correlation function of x
n	Noise waveform, eqn. 22
u,v,w	Signal modulations
x	Random phase angle
Star	On the line - convolution
	Superfixed - complex conjugate
τ, σ	Variables of integration (time)
ψ	Auto-correlation function of n, eqn. 24

1. Introduction

Since originally proposed as a useful description of the properties of a radar modulation (ref. 1), the ambiguity function has been subject to considerable study and further development (refs. 3-8), but it is not an easy function to compute. This paper sets out the mathematics upon which some simple constructions may be based and derives formulae and schematic diagrams for certain primitive modulations. Most of these are of random or quasi-random type, being the simplest forms of high resolution signal to treat. They include, as a particular case, the commonest of all modulations, the incoherently phased pulse-train, reached in section 11 of the paper.

As a summary of the definitions from which we start, we have in the first place the generalized auto-correlation function of any complex modulation u(t). This is given by

$$(1a) \qquad A(t,f) = \int_{-\infty}^{\infty} u^*(p-\tfrac{1}{2}t)\,u(p+\tfrac{1}{2}t)\,e^{-2\pi i f p}\,dp$$

or the equivalent formula

$$(1b) \qquad A(t,f) = e^{-\pi i f t}\int_{-\infty}^{\infty} u^*(p)\,u(p+t)\,e^{-2\pi i f p}\,dp$$

which is a complex function of time t and frequency f, though in simple cases it often turns out to be purely real. The function describes the characteristic response of an ideal signal filtering system designed to indicate the time-shift t and frequency-shift f imposed on a transmitted radar signal as a result of echoing from a point target. Its form is characteristic of the radar modulation u(t), and ideally it would be zero at all points in the t-f plane except at the origin, which corresponds to the target position. Thus a target at (T, F) would produce a response A(t-T, f-F), but in reality the diffuse form of A produces indications not only at (T, F) but at many other points besides. The resulting ambiguity of radar measurement is mathematically unavoidable, however the received signal may be processed, because the ambiguity resides fundamentally in the transmitted modulation.

The use of complex functions is apt to cause confusion in our minds, but is essential to the mathematical treatment. Some reminders may therefore be not out of place. The actual voltage waveform transmitted is assumed to have the form

$$\text{Real part of} \quad u(t)\,e^{2\pi i f_0 t}$$

where f_0 is the radio frequency and u(t) is a low-frequency modulation, real for pure amplitude modulation and complex for phase modulation. The voltage response from the filtering system is given by

$$\text{Real part of} \quad A(t,f)\,e^{2\pi i f_0 t}$$

before detection. After detection, it would become

$$D(|A(t,f)|)$$

where D represents the detector characteristic. When considering interference between signals from separate targets, strict analysis demands addition of A's, shifted appropriately in t and f and properly phased in relation to each other. If, however, the density of targets is so great that the number of (effectively randomly phased) superpositions is large, as in the case of clutter, the appropriate function for convolution as pointed out by G. R. Whitfield (ref. 9) is

$$|A(t,f)|^2.$$

Clutter apart, this function is fundamental in the mathematical theory for a single-point target, and is known as the unnormalized ambiguity function. Its importance is due to its two properties (ref. 1)

$$(2) \qquad |A(t,f)|^2 \leq A^2(0,0)$$

$$(3) \qquad \iint_{-\infty}^{\infty} |A(t,f)|^2 \, dt \, df = A^2(0,0).$$

When normalized, the ambiguity function has the form

$$\frac{|A(t,f)|^2}{A^2(0,0)}$$

which can be regarded as the actual degree of ambiguity on a scale from 0 to 1, between two signals which differ from each other through a time-shift, t, and a frequency-shift, f. The integral of the normalized ambiguity function with respect to t and f is always equal to unity, from (3). Thus, in view of (2), the ambiguity of a modulation must be spread over unit area of the t-f plane at the very least. Random modulation and other complicated waveforms are mostly aimed at reducing unwanted ambiguity by spreading it more thinly over much larger areas.

2. Siebert's Theorem

The definition of A(t, f) is more symmetrical in t and f than might appear from (1a). An alternative form (ref. 1) is

$$(4) \qquad A(t,f) = \int_{-\infty}^{\infty} U(p+\tfrac{1}{2}f)\, U^*(p-\tfrac{1}{2}f)\, e^{2\pi i p t}\, dp$$

where U(f) is the Fourier Transform of u(t). The two forms (1a) and (4) when considered together, show close symmetry in time and frequency. Further, Siebert's Theorem (ref. 3) states that

$$(5) \qquad \iint_{-\infty}^{\infty} |A(T,F)|^2 e^{2\pi i (fT - Ft)}\, dT\, dF = |A(t,f)|^2$$

The ambiguity function is its own two-dimensional Fourier Transform with time and frequency interchanged.

3. The single pulse

The whole of the present treatment is restricted, for reasons of simplicity, to the Gaussian function in one shape or form. There is an immediate loss of generality but a very great simplification, enabling one to draw rule-of-thumb conclusions of quite wide applicability when refined calculations are not required. In keeping with this approach, we define the simplest radar pulse modulation as

$$(6) \qquad v(t) = k.e^{-at^2}$$

which is a single (physically unrealisable!) pulse centred at $t = 0$. By substituting into (1) and choosing the normalising constant as

$$k = \left(\frac{2a}{\pi}\right)^{1/4}$$

we obtain

$$(7) \qquad A(t,f) = e^{-\frac{1}{2}at^2} e^{-\frac{\pi^2 f^2}{2a}}$$

and hence the (normalised) ambiguity function

$$(8) \qquad |A(t,f)|^2 = e^{-at^2} e^{-\pi^2 f^2/a}$$

which is an elliptical Gaussian hill in t and f. Conveniently we may define the "width" of a Gaussian function as the product of standard deviation with $\sqrt{(2\pi)}$, because this makes the product of widths of $v(t)$ and its spectrum equal to unity (see Fig. 1). Thus, by definition, the signal duration and bandwidth become

$$(9) \qquad \begin{aligned} T &= \sqrt{\pi/a} \\ W &= \sqrt{a/\pi} \end{aligned}$$

These two quantities happen, in this simple instance, to be the widths of the ambiguity function along its principal axes. It may at first seem surprising that no factor of root two has crept in, since ambiguity is a squared and therefore a narrowed quantity, but the narrowing is exactly balanced by the broadening due to auto-correlation.

Schematically, we may represent the basic Gaussian ambiguity pattern (8) by drawing a rectangle as in Fig. 2. We may conveniently think of this as an "equivalent" ambiguity surface, distorted geometrically but indicating widths and volumes correctly. The diagram represents a brick-shaped hill of unit altitude where shaded black, zero elsewhere, and of unit volume as $WT = 1$.

The single pulse is the most basic form of radar with <u>minimal</u> resolution in time and frequency.

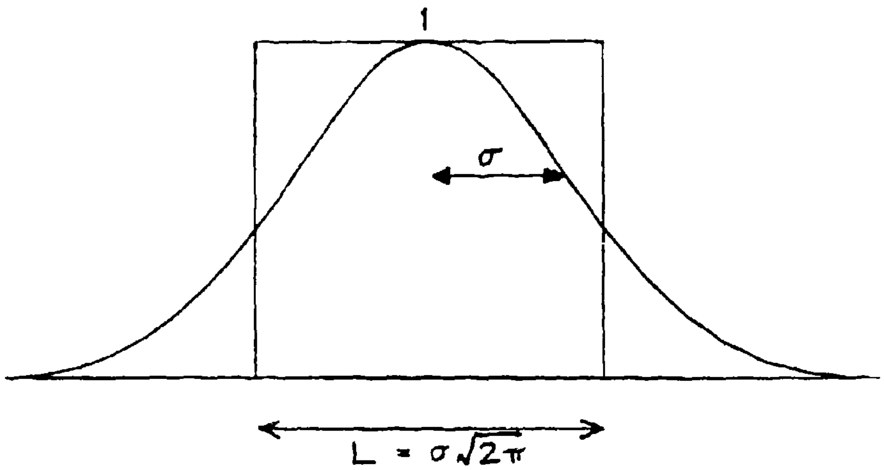


Fig 1. Gaussian function
L = width used throughout the paper

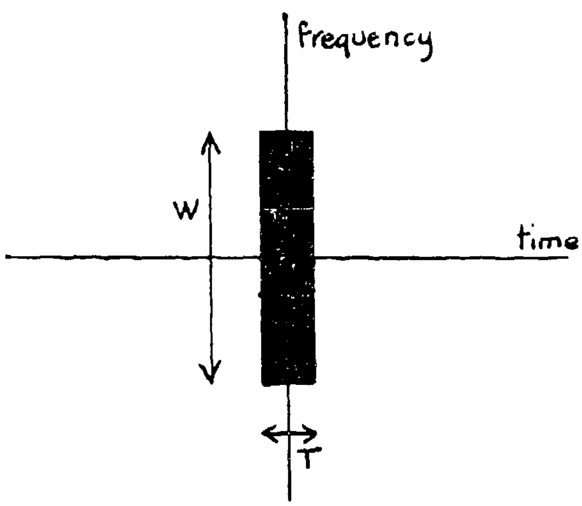


Fig 2. Schematic representation of Gaussian ambiguity function G(T, W) for a single radar pulse of duration T and bandwidth W.

For this simple pulse, TW = 1.

This and all subsequent Figures are drawn to same scale.

4. The "bed of nails"

One of the most useful idealized waveforms in radar theory is a train of unit delta-functions, which has the convenient property of Fourier transforming into itself when the pulse repetition period is taken to be unity. From this waveform any periodic function can immediately be generated by scaling and convolution. It has been shown elsewhere that such a waveform gives rise to a generalized auto-correlation function of the form

$$A(t,f) = \sum_n \sum_m \delta(t-n)\,\delta(f-m) \tag{10}$$

which has been described by Robert Price (ref. 6) as a "bed of nails" - sharp side up.

It is perhaps fortunate that questions of normalization need not concern us unduly, as they generally become clear when physical realism has been imposed. The function (10) would be particularly troublesome in this respect, as the normalization factors associated with it are always zero or infinite. For example, the radar modulation leading to (10) would consist of an infinitely long train of pulses each of zero energy, finite amplitude and zero duration. And the associated normalized ambiguity consists of an infinite number of packets of zero ambiguity at the integer lattice points in t and f. These physical absurdities are unembarrassing, as the bed of nails is merely a means to an end.

Simple changes of variable show that the lattice spacings can be generalized, as in Fig. 3, to become R in time and 1/R in frequency, where R is any arbitrary repetition period. The modulation giving rise to such a lattice diagram is of course the pulse-train

$$u(t) = k \sum_n \delta(t-nR) \tag{11}$$

In the limits $R = 0$ and $R = \infty$, the ambiguity diagram degenerates. In one direction all but a single row of nails pass out of the picture, whilst at the same time in the other direction the nails crowd together into what might be described as a razor edge. This is also the limit of the single pulse considered in section 3 when either W or T goes to zero.

The modulation (11) is bedevilled by periodic ambiguity, but apart from this the resolution in t and f is perfect, except in the limits just mentioned. When in these limits the nails make contact, the modulation suddenly becomes an example of a radar with minimal resolution.

5. Convolution of modulations

Let v(t) and w(t) be two complex modulations, and let A_v and A_w be their auto-correlation functions. Let u(t) be the convolution of v and w, thus

$$u(t) = \int_{-\infty}^{\infty} v(x)\, w(t-x)\, dx. \tag{12}$$

We may wish to know the auto-correlation function of u. Substituting in (1a), we obtain

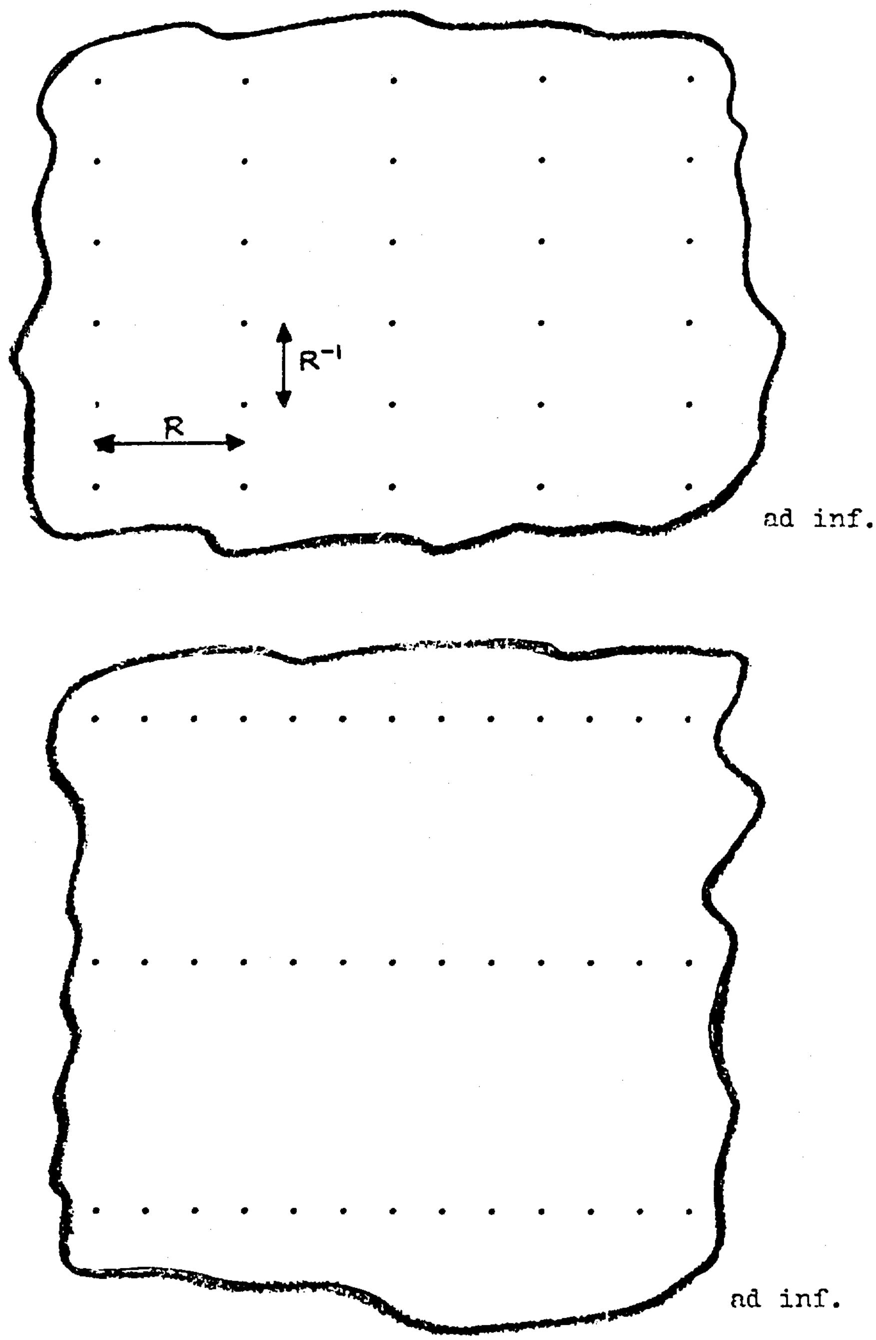


Fig 3. Delta-lattice or "bed of nails" $I(R, R^{-1})$

$$A_u = \iiint v^*(x)\,v(y)\,w^*(p-\tfrac{1}{2}t-x)\,w(p+\tfrac{1}{2}t-y)\,e^{-2\pi i f p}\,dx\,dy\,dp$$

Simple manipulations and changes of variable enable us to recast this expression in the form

$$A_u(t,f) = \int_{-\infty}^{\infty} A_v(X,f)\,A_w(t-X,f)\,dX, \tag{13}$$

which is the convolution of A_v and A_w with respect to the time-dimension. An exactly similar result can be derived in the frequency dimension. Thus we have two useful rules:

Rule 1 If two modulations are convolved in the time domain (or multiplied in the frequency domain), their generalized auto-correlation functions are convolved in the time coordinate.

Rule 2 If two modulations are convolved in the frequency domain (or multiplied in the time domain), their generalized auto-correlation functions are convolved in the frequency coordinate.

The operation of these two rules is non-commutative.

6. Functional notation

We now set up a functional notation adapted to the application of rules 1 and 2 and designed to facilitate the process of transferring mathematical formulae to graphical form. The notation is not always well adapted to detailed analytical work, such as the evaluation of integrals, and will be brought into play only when appropriate.

(a) Gaussian function of one variable G(L)

For reasons which are not merely obtuse, the variable itself is not stated. G(L) stands for a gaussian function of width L, centred at the origin, "width" being defined by

$$L = \sigma\sqrt{2\pi} \tag{14}$$

where σ is the standard deviation. The amplitude is taken to be unity in the middle. Thus, for example, the waveform

$$u(t) = e^{-\pi(t/T)^2}$$

would be written

$$= G(T).$$

It is particularly to be noticed that the Fourier Transform of this waveform u, defined as usual with conjugate variables t (seconds) and f (cycles per second), is simply

$$U = T.G(T^{-1})$$

(b) Gaussian function of two variables G(L, L')

This denotes the product of G(L) in one variable with G(L') in

another. Throughout this paper the two variables are understood to be t and f respectively.

(c) Row of nails in one variable $I(R)$

A train of unit delta functions at regular separations R. For example, equation (11) would be written as

$$u = k.\,I(R)$$

(d) Bed of nails in two variables $I(R, R')$

Consists of unit delta-functions at the lattice points (nR, mR') where n and m are integers. For example, with variables t and f understood, equation (10) would be written as

$$A = I(1,1).$$

The more general case illustrated in Fig. 3 is

$$A = I(R, R^{-1})$$

(e) Convolution

A letter t or f under the star denotes the variable of convolution. For example, equation (13) would be written

$$A_u = A_v \underset{t}{*} A_w$$

(f) Arbitrary normalizing constant k

It is convenient to reserve a letter, such as k in this paper, for a constant whose value need not be consistent in the analysis.

7. The coherent pulse-train

The simplest finite pulse-train for mathematical analysis consists of the carrier modulated in amplitude by two functions together, a succession of gaussian shaped pulses and an overall gaussian taper long compared with the repetition rate. This may be written in the notation of the previous section as

$$(15) \qquad k.\,(G(W^{-1}) * I(R)).\,G(T)$$

where
R = repetition period
T = duration of train
W = bandwidth
$WT >> 1$

Before we can apply either of rules 1 and 2, we require the generalised auto-correlation function of G. In section 3, it has been stated that the waveform (6) gives the auto-correlation function (7). Rewriting in terms of the widths defined at (9), we have the easily proved result that the waveform

$$(16) \qquad u = k.\,G(T)$$

has the generalized auto-correlation function

$$A = G(T\sqrt{2}, T^{-1}\sqrt{2}) \tag{17}$$

Applying rules 1 and 2 of section 5, we thus obtain the auto-correlation function of (15) in the form

$$A = (G(W^{-1}\sqrt{2}, W\sqrt{2}) \underset{t}{*} I(R, R^{-1})) \underset{f}{*} G(T\sqrt{2}, T^{-1}\sqrt{2}) \tag{18}$$

and, after squaring,

$$|A|^2 = (G(W^{-1}, W) \underset{t}{*} I(R, R^{-1})) \underset{f}{*} G(T, T^{-1}) \tag{19}$$

which is illustrated in Figure 4, a well-known diagram. It should particularly be noted that the central zone, which is a measure of the radar resolution, can be made as fine as desired by increasing the duration T and bandwidth W of the radar transmission, assuming that both are fully utilized.

8. Complex noise modulation

The ambiguities shown in Figure 4 may prove embarrassing in a practical situation. They can be smoothed out, while preserving the central zone, as illustrated in Figure 5, if the radar transmits noise modulation of the same duration and bandwidth as before. This represents a Gaussian hill of unit amplitude standing on a lower wider Gaussian pedestal; it is the aim in this section of the paper to derive and comment upon this rather well-known result.

First it is necessary to take note of the statistical nature of the ambiguity function. Any particular high-resolution noise waveform possesses a normalized ambiguity function whose value is unity at the origin, and has a random skirt, pedestal, or side-lobe pattern, extending well beyond the central zone. A different sample of noise would produce a different ambiguity function, the ups and downs of the pedestal differing in their detailed arrangement. The only simple answer is to work with the mean ambiguity function for an ensemble of noise modulations, i.e. the mean squared auto-correlation function

$$\overline{|A(t,f)|^2} \tag{20}$$

This proves to be unnecessarily restrictive in the analysis, and instead of it we propose to study the function

$$\overline{A(t_1, f_1)\, A^*(t_2, f_2)} \tag{21}$$

which is the complex co-variance of the generalized auto-correlation function. By putting the two t's and the two f's equal, the mean ambiguity is immediately obtainable, but (21) can tell us more. It contains information about the structure of the random variations of ambiguity in the pedestal. When (21) is zero, the fluctuation of ambiguity at (t_1, f_1) can be regarded as uncorrelated with that at (t_2, f_2). It may be important to know about this when a

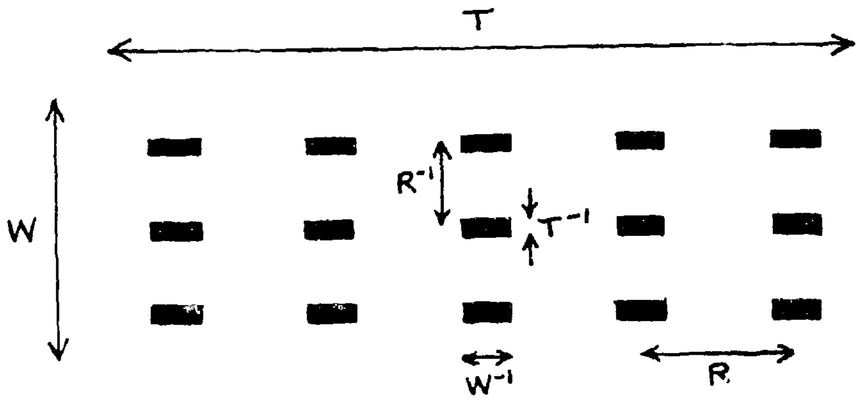


Fig 4. Ambiguity of coherent pulse-train

T = duration of train R = repetition period
W = bandwidth

The above diagram is drawn with TW = 15

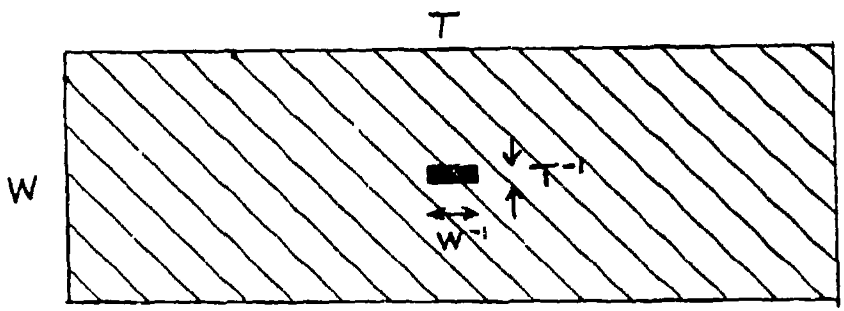
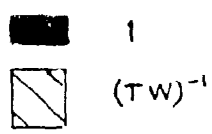


Fig 5. Ambiguity Z(T, W) of a burst of noise

T = duration of burst W = bandwidth of noise



convolution is performed upon A, since simple addition in "power" is valid only in the absence of correlation.

The modulation to be considered in this section will be of the form

$$u(t) = v(t) . n(t) \tag{22}$$

where v(t) is Gaussian amplitude taper (cf. eqn (6))

$$v(t) = +\sqrt{\frac{2a}{\pi}}\, e^{-at^2} \tag{23}$$

and n(t) is a complex stationary random function of time. Its power spectrum will be assumed to be centred upon zero frequency, to be of Gaussian shape, and indeed n(t) will be assumed to have all the usual properties of Gaussian noise. Thus, when modulating the radio-frequency carrier, n(t) would produce a typical narrow-band noise waveform displaying random fluctuations both in amplitude and phase. (Had we postulated that n(t) be purely real, this would not have been the case, and the radar signal would have been modulated in amplitude only).

To put a scale on the time-structure of n(t), equivalent to setting its bandwidth, we shall now assume the following auto-correlation function (for comment, see Appendix 2)

$$\psi(t) = \overline{n(\tau)\, n^*(\tau + t)} = e^{-bt^2}. \tag{24}$$

In a statistical sense, the waveform u(t) is now fully determined. The constants have been chosen so as to make

$$\int_{-\infty}^{\infty} \overline{|u|^2}\, dt = \overline{|n|^2} \int_{-\infty}^{\infty} v^2(t)\, dt = 1 \tag{25}$$

The constants a and b control respectively the duration of the noise burst and the correlation time of the noise.

Evaluation of (21), using the definition (1b), proceeds as follows

$$\begin{aligned}
&\overline{A_1 A_2^*} \\
&= \overline{A(t_1, f_1)\, A(t_2, f_2)^*} \\
&= e^{-\pi i (f_1 t_1 - f_2 t_2)} \iint v(\tau)\, v(\tau + t_1)\, v(\sigma)\, v(\sigma + t_2) \;\times \\
&\quad \overline{n^*(\tau)\, n(\tau + t_1)\, n(\sigma)\, n^*(\sigma + t_2)}\, .\; e^{-2\pi i (f_1 \tau - f_2 \sigma)}\, d\tau\, d\sigma
\end{aligned} \tag{26}$$

It is shown in Appendix 2 that the system averaged quadruple product is equal to

$$\psi(t_1)\psi(t_2) \quad + \quad \psi(\tau-\sigma)\psi(\tau-\sigma+t_1-t_2). \tag{27}$$

Substituting into (26), we find that the first of these terms leads to a separate product of single integrals, which can be identified as

$$\overline{A_1}\,\overline{A_2^*} \quad = \quad e^{-\pi i(f_1 t_1 - f_2 t_2)} \quad \times$$
$$\psi(t_1)\int v(\tau)v(\tau+t_1)e^{-2\pi i f_1 \tau}d\tau .\ \psi(t_2)\int v(\sigma)v(\sigma+t_2)e^{2\pi i f_2 \sigma}d\sigma \tag{28}$$

This is the "systematic" contribution to the mean squared modulus of A, reducing to the square of the mean, and it corresponds, as we might anticipate, to the central zone of the ambiguity diagram. The integrals are readily evaluated, using (6) and (7) as the pattern; using (24) also, we obtain

$$\overline{A_1}\,\overline{A_2^*} \quad = \quad e^{-\frac{1}{2}(a+2b)(t_1^2+t_2^2)}\ e^{-\frac{\pi^2}{2a}(f_1^2+f_2^2)}. \tag{29}$$

Hence, identifying suffixes 1 and 2, we find

$$|\overline{A}|^2 \quad = \quad G(W^{-1}, T^{-1}) \tag{30}$$

where T and W are given by

$$T \ = \ \text{duration of noise burst} \ = \ \sqrt{\pi/a} \tag{31}$$

$$W \ = \ \text{bandwidth of noise} \ = \ \sqrt{(a+2b)/\pi} \tag{32}$$

The two terms contributing to W arise because the amplitude modulation of the noise broadens the bandwidth slightly. In an application the condition

$$b >> a \qquad (\text{i.e. } TW >> 1) \tag{33}$$

would be likely to apply, and the spectral broadening due to the Gaussian taper would be very slight.

The second term of (27) also contributes to (26) and the resulting double integral does not separate so readily. To describe the evaluation as merely tedious would be an understatement. The result, however, is simple to write down, and is

$$(34) \qquad \sqrt{a/(a+2b)}\; e^{-\frac{1}{2}a(t_1^2+t_2^2)}\, e^{-\frac{1}{2}b(t_1-t_2)^2} \times$$
$$e^{-\frac{\pi^2}{2(a+2b)}(f_1^2+f_2^2)}\, e^{-\frac{\pi^2 b}{2a(a+2b)}(f_1-f_2)^2}$$

Hence, by equating suffixes 1 and 2, and remembering to include the term (29) already derived, we obtain

$$(35) \qquad \overline{|A|^2} = e^{-t^2(a+2b)} e^{-\pi^2 f^2/a} + \sqrt{\frac{a}{a+2b}} \cdot e^{-at^2} e^{-\pi^2 f^2/(a+2b)}$$

or, in functional form, using (31) and (32),

$$(36) \qquad \overline{|A|^2} = G(W^{-1}, T^{-1}) + (TW)^{-1} G(T, W) = Z(T, W) \text{ say}$$

This is the result illustrated in Figure 5. It shows that ambiguity is spread with density $1/WT$ over a region corresponding to the product of the duration and bandwidth of the signal, whilst the central zone, which is exactly similar in shape, has a size governed by the reciprocals of signal bandwidth and duration. In the limit when $T = W = \infty$, this radar is perfect.

An interesting question is whether (36) can be said to be properly normalized. When no random element is present, the normalized value of A and hence also of its square, is unity at $t = f = 0$. This can be assured from equations (1) by normalizing the radar modulation so that

$$(37) \qquad \int_{-\infty}^{\infty} |u(t)|^2 dt = 1.$$

When there is a random element in the waveform, equation (25) merely ensures

$$\overline{|A(0,0)|} = 1,$$

but does not ensure

$$\overline{|A(0,0)|^2} = 1.$$

Indeed, from equation (36), for example, we have

$$(38) \qquad \overline{|A(0,0)|^2} = 1 + (TW)^{-1}$$

The small second term (cf. (33)) is due to the random pedestal upon which the normalized central zone stands, and is the mean squared deviation of A from its squared mean. However, the right hand side of (38) can also be interpreted, from the theorem of equation (3), as the total volume under the ambiguity surface. It is now clear that the first term on the right of (38) arises from the integral of the <u>second</u> term of (36), which is the mean volume of the pedestal, while the smaller second term in (38) is the volume of the central zone. Thus neither of the terms in (38) can be associated uniquely with one or other of the components of the ambiguity function (36). To

return to the question of normalization, it is clear that the fully normalized function would be

$$\text{(39)} \qquad \frac{TW}{1+TW}\, G(W^{-1}, T^{-1}) + \frac{1}{1+TW}\, G(T, W)$$

but there is no special requirement for this form.

As for the fluctuations of the pedestal, inspection of (34) shows that these are effectively de-correlated at separations given by

$$\text{(40)} \qquad (t_1 - t_2)^2 = 2\pi/b$$

$$\text{(41)} \qquad (f_1 - f_2)^2 = 2a(a+2b)/(\pi b)$$

i.e. at separations which are similar in magnitude and direction to the radius vector of the limit to the central zone of the diagram. This would be expected on purely intuitive grounds, as the central zone describes the typical resolution of the matched filter outputs.

It is interesting to note that the central zone is the two-dimensional Fourier Transform of the mean pedestal and vice versa. This is evident from Siebert's Theorem, and it must generally ensure that the central zone is similar in shape to the outer boundary of the ambiguity diagram.

9. Bursts of complex noise

By applying rules 1 and 2 of section 5, the basic ambiguity diagram derived in the previous section for a single noise burst can be generalized for more complicated signals. In this section we consider two cases

(i) a finite train of identical complex noise bursts

(ii) a finite train of independent complex noise bursts

In case (i), it is necessary to qualify the word "identical", as the limitation on the number of bursts will be imposed by means of a slow Gaussian amplitude taper extending over an otherwise infinite train. Let

(42) W = bandwidth of the noise

T = duration of the train

P = duration of each burst

R = repetition period between bursts.

For the present, we shall assume that

$$\text{(43)} \qquad PW \gg 1.$$

Consider case (i). The passage from a single burst to a train of identical bursts is exactly similar to the discussion at equations (16)-(19) applying to an ordinary pulse.

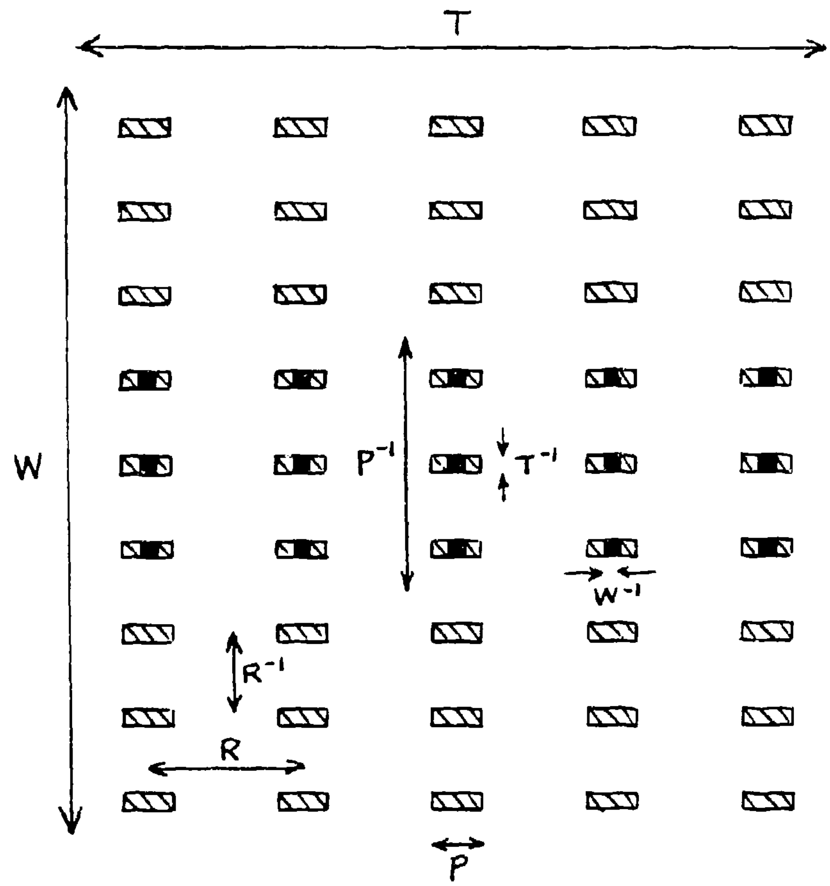
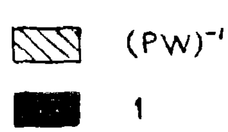


Fig 6. Ambiguity of coherent train of noise-bursts

T = total duration of signal W = bandwidth
P = duration of each burst R = repetition period

Each burst is made from the same noise pattern



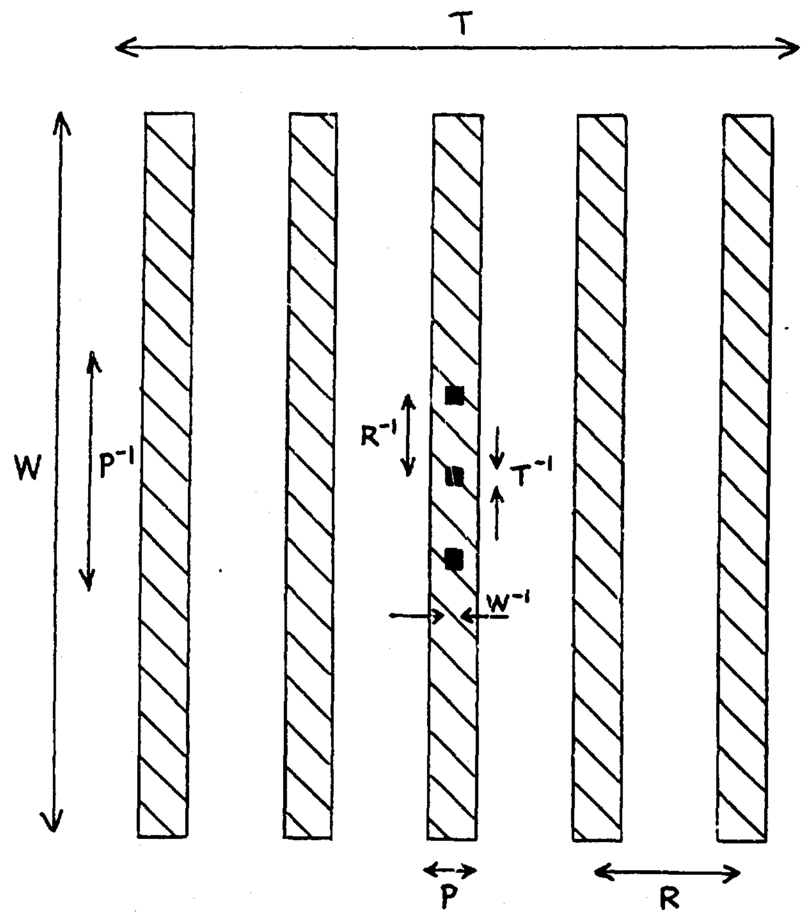
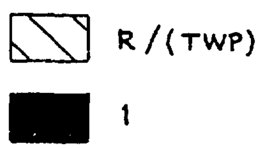


Fig 7. Ambiguity of train of independent noise bursts

T = total duration of signal W = bandwidth
P = duration of each burst R = repetition period



Hence the ambiguity function is

$$(44) \qquad (Z(P,W) \underset{t}{*} I(R,R^{-1})) \underset{f}{*} G(T,T^{-1})$$

where Z is the ambiguity function for a single burst, i.e.

$$(45) \qquad Z(P,W) = G(W^{-1},P^{-1}) + (PW)^{-1}G(P,W)$$

from equation (36). The function (44) is shown at Fig. 6. At those lattice points which are not, to a crude approximation, fully ambiguous, the density of ambiguity is found to be $1/WP$, as also in the wings of all the ambiguities.

In case (ii), it is necessary to give the single burst which we take as our starting point a duration T, and then to multiply this by a modulation

$$(46) \qquad G(P) * I(R)$$

which is an infinitely long train of amplitude pulses each of length P and separation R. Hence the ambiguity function is

$$(47) \qquad Z(T,W) \underset{f}{*} (G(P,P^{-1}) \underset{t}{*} I(R,R^{-1}))$$

as illustrated in Figure 7. On this diagram, the bandwidth marked W is slightly greater than the W appearing in the expression (47), owing to the effect of convolution in frequency. In other words, the resultant bandwidth has been slightly broadened by the amplitude modulation associated with P, which (unlike that due to T) is not included in the W defined at (32) as used in the formula (47).

For completeness, and for subsequent reference, a further ambiguity diagram can be deduced from the last expression. Let us retain the original definition of W given at (32), but now choose P so small as to make

$$PW < 1$$

i.e. let the individual pulses be so short that there is no time for the noise to fluctuate within the pulse. The signal is then a pulse-train whose bandwidth is entirely controlled by P. It is noise-modulated only inasmuch as each individual pulse possesses a random phase and amplitude, and the ambiguity diagram will be found to be as shown at Figure 8. A modulation of this type is, of course, never used in a practical radar and is of purely theoretical interest.

10. Noise amplitude modulation

A radar signal endowed with random amplitude modulation but no phase modulation is a somewhat artificial idea, but it possesses mathematical interest and is included here for the sake of generality. The amplitude modulation is of the type which permits positive and negative amplitudes (in fact a Gaussian distribution) and hence introduces phase-reversals of an artificial character.

The analysis starts from equation (26) as before, though the conjugate signs are now redundant both in that equation and in the definition (24). As is clear from Appendix 1, an additional term must now be included in (27). This is a term

(48) $$\psi(t_1+\tau-\sigma)\psi(t_2+\sigma-\tau)$$

Putting $t_1+\tau=\tau'$, the additional contribution to (26) may be written in full as

$$e^{\pi i(f_1t_1+f_2t_2)}\iint v(\tau'-t_1)v(\tau')v(\sigma)v(\sigma+t_2)\,.$$
$$\psi(\tau'-\sigma)\,\psi(\tau'-\sigma-t_1-t_2)\,e^{-2\pi i(f_1\tau'-f_2\sigma)}\,d\tau'\,d\sigma$$

and, by inspection, we see that this is exactly the same as the contribution to (26) from the second term of (27) with the sign of t_1 changed. Consequently, an extra term must be added to (35) by changing the sign of t_1 in (34), i.e.

(49) $$e^{-\frac{1}{2}b(t_1-t_2)^2}$$

in (34) must be replaced by

(50) $$e^{-\frac{1}{2}b(t_1-t_2)^2}+e^{-\frac{1}{2}b(t_1+t_2)^2}.$$

Upon equating the suffixes 1 and 2, the resulting ambiguity function is found to be

(51) $$\overline{|A|^2} = e^{-t^2(a+2b)}e^{-\pi^2f^2/a}$$
$$+\left(\frac{a}{a+2b}\right)^{1/2}e^{-\frac{\pi^2f^2}{a+2b}}e^{-at^2}$$
$$+\left(\frac{a}{a+2b}\right)^{1/2}e^{-\frac{\pi^2f^2}{a+2b}}e^{-t^2(a+2b)}$$

or, in functional form, using (31) and (32),

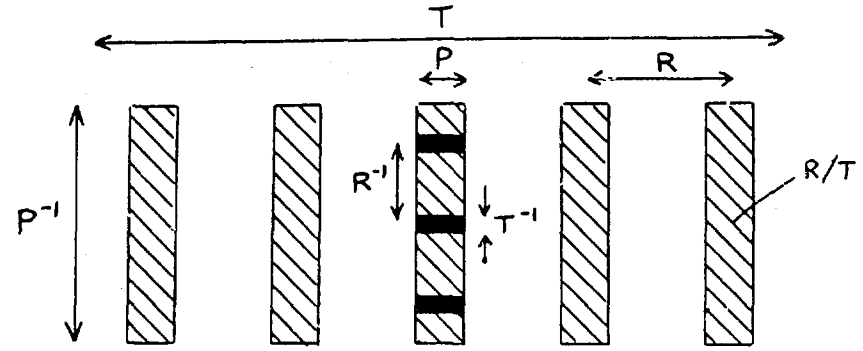


Fig 8. Ambiguity of train of uncorrelated pulses

P = pulse length T = duration of train
R = repetition period

Each pulse has an independent random amplitude & phase

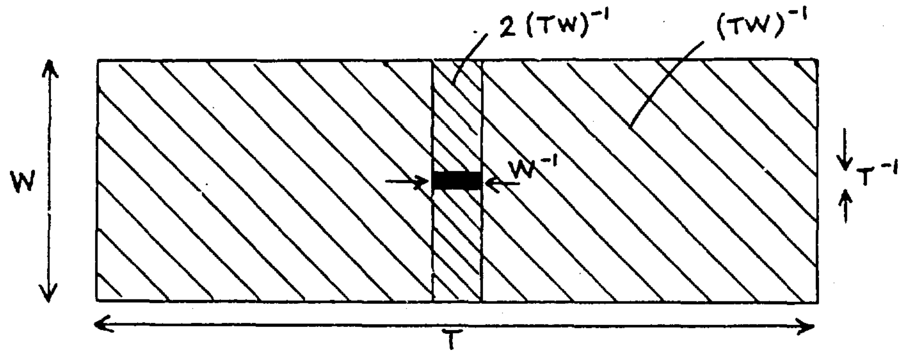


Fig 9. Ambiguity X(T, W) of a burst of noise amplitude modulation

T = duration of burst W = bandwidth

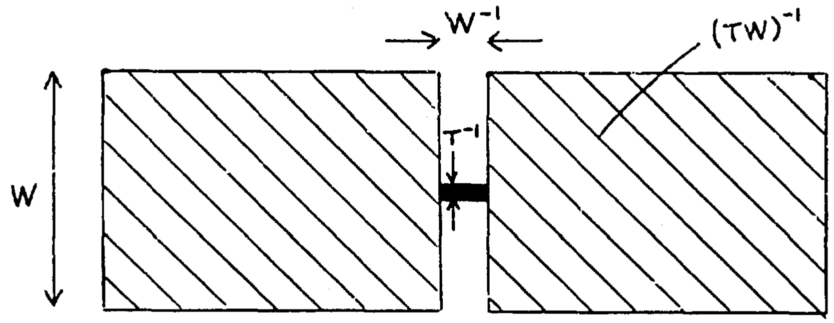


Fig 10. Ambiguity Y(T, W) of a burst of random phase modulation

Note the clear central strip

$$(52) \quad \overline{|A|^2} = G(W^{-1}, T^{-1}) + (TW)^{-1} G(T, W) + (TW)^{-1} G(W^{-1}, W)$$

$$= X(T, W), \text{ say}$$

as illustrated in Figure 9. The analysis is exact.

That amplitude noise modulation suffers in comparison with complex noise only in its ability to resolve Doppler when there is no range difference must presumably be due to the excessive fades in an amplitude-modulated signal of the type considered. When there is a quadrature component, these fades are, for the most part, filled in.

Mathematically, the result is interesting for two reasons. If the parameter b is set equal to zero, the noise will have no time to fluctuate before the burst is over. Hence the signal will consist of a simple Gaussian-shaped pulse of Gaussian randon amplitude, an extreme form of non-ergodic modulation. From (31) and (32) we see that WT is then equal to unity as we should expect. However, we find from (52)

$$(53) \quad (\bar{A})^2 = G(T, T^{-1}) \qquad b = 0$$

$$(54) \quad \overline{|A|^2} = 3\, G(T, T^{-1}) \qquad b = 0$$

By contrast, for a complex noise-burst which lacks the third term in (52) we have

$$(55) \quad \overline{|A|^2} = 2\, G(T, T^{-1}) \qquad b = 0$$

These results are due to the fact that, for a real Gaussian random number n, we have the well-known property

$$(56) \quad Av(n^4) = 3\left[Av(n^2)\right]^2$$

whilst for a complex random Gaussian number, we have

$$\begin{aligned} Av\,|n^4| &= Av\left[(x^2+y^2)^2\right] \\ &= Av(x^4) + 2\,Av(x^2)Av(y^2) + Av(y^4) \\ &= (3+2+3)\left[Av(x^2)\right]^2 \\ &= 2\left[Av(x^2+y^2)\right]^2 \\ (57) \quad &= 2\left[Av\,|n^2|\right]^2 \end{aligned}$$

which acts as a crude check on the results so far obtained.

The second point of interest lies in comparing the ambiguity function for amplitude modulation with that for pure phase modulation, to which we now pass.

11. Random phase modulation

We now assume a modulation of the form given at equation (22), with n(t) re-defined to be

$$n(t) = e^{-ix(t)} \tag{58}$$

where x(t) is a purely real Gaussian random function. This will be a noise-burst, smooth and gaussian-shaped in amplitude but randomly varying in phase. The desired co-variance of auto-correlation (26) will then entail evaluation of the system average of the form

$$M = \overline{\exp\left(ix_1 - ix_2 - ix_3 + ix_4\right)} \tag{59}$$

in place of the barred product in (26). Fortunately this is directly obtainable from the moment generating function defined at (1-2) in Appendix 1.

Putting $y_1 = y_4 = 1$ and $y_2 = y_3 = -1$, we have

$$M = e^{-\frac{1}{2}Q} \tag{60}$$

where

$$\begin{aligned} Q = \overline{x_1^2} &- 2\overline{x_1x_2} - 2\overline{x_1x_3} + 2\overline{x_1x_4} \\ &+ \overline{x_2^2} + 2\overline{x_2x_3} - 2\overline{x_2x_4} \\ &+ \overline{x_3^2} - 2\overline{x_3x_4} \\ &+ \overline{x_4^2} \end{aligned} \tag{61}$$

Here we have used the shorthand notation

$$\begin{aligned} x_1 &= x(\tau) \\ x_2 &= x(\tau + t_1) \\ x_3 &= x(\sigma) \\ x_4 &= x(\sigma + t_2) \end{aligned} \tag{62}$$

Now denoting the auto-correlation function of x by c, we have from (61) and (62)

$$M = e^{-2c(0) + c(t_1) + c(t_2) + c(\tau-\sigma) + c(\tau-\sigma+t_1-t_2) - c(\tau-\sigma-t_2) - c(\tau-\sigma+t_1)} \tag{63}$$

To obtain the ambiguity function, it is necessary to make approximations, dividing the t-f plane into two regions now to be considered in turn.

(i) t_1 and t_2 small

When the t's are zero, the exponent in (63) vanishes entirely, we have $M = 1$, and the whole problem becomes identical to an unmodulated burst, i.e. to the simple pulse considered in section 3. This provides the clue for approximation when the t's are small but not necessarily zero, which is to expand the functions appearing in (63) in the form of power series and see what happens. Auto-correlation functions are always even and may be expanded in the form

(64) $$c(t) = a_0 - a_2 t^2 + \cdots$$

More appropriately we may write

(65) $$c(t) = c(0).\left(1 - 2\pi^2 B^2 t^2 + \ldots\right)$$

where B, which will not really enter the present discussion, is interpreted by the well-known relation

(66) $$B^2 = \text{variance of power spectrum of } x(t) \text{ in cps}^2$$

Expansion of all the terms in (63) leads to so much cancellation that all terms in τ and σ vanish, and we are left with

(67) $$M = e^{c(t_1) + c(t_2) - 2c(0)}$$

which comes outside the integrals in (26). To interpret this result, we must relate the auto-correlation function c, which applies to the random phase angle, to the auto-correlation function for the actual noise modulation. At (24) we assumed

(68) $$\psi(t) = \overline{n(\tau)\, n^*(\tau + t)} = e^{-bt^2}$$

Substituting from (58), we now find

(69) $$\psi(t) = \overline{\exp\left(-ix(\tau) + ix(\tau + t)\right)}$$

and this can be evaluated in a similar manner to (59). Putting $y_3 = y_4 = 0$ in (A-2), we see that

(70) $$\psi(t) = e^{c(t) - c(0)}$$

and hence (67) becomes

(71) $$M = \psi(t_1)\psi(t_2) = e^{-b(t_1^2 + t_2^2)},$$

the last equation coming from (68). Identifying suffixes 1 and 2,

this expression can be introduced as a multiplying factor to the result for an unmodulated burst given at (8). So, finally,

$$\overline{|A|^2} = e^{-(a+2b)t^2} e^{-\pi^2 f^2/a}$$

$$= G(W^{-1}, T^{-1}) \qquad (72)$$

exactly as at (30). The significance of this result is considerable, since the previous result for complex noise at (30) was only a part of that answer. The ordinary complex noise burst has, in addition, a pedestal term which is, in the present case, entirely absent. There is, in fact, a clear strip along the f-axis outside the central zone given by (72). This is fully in accordance with simple engineering commonsense, since the random phase modulation does nothing to impede a good Doppler determination provided that there is no unknown time-delay to produce a statistical muddle from temporal auto-correlation. If the range of the target is known, the randomness of the phase modulation is entirely irrelevant and compensatable.

(ii) t_1 and t_2 large

When the t's are large compared with the width of c(t), an entirely different method of evaluating (63) is necessary, because the power series expansions of sub-section (i) cease to be valid. The trick is to regard the exponent in (63) as a function of τ and σ, or more simply as a function of $\tau - \sigma$, and consider the effect of integration with respect to τ and σ. The exponent is made of the terms

(a) $-2c(0)$

(b) $+ c(t_1) + c(t_2)$

(c) $- c(\tau-\sigma+t_1) - c(\tau-\sigma-t_2)$

(d) $+ c(\tau-\sigma) + c(\tau-\sigma+t_1-t_2)$

For large t, auto-correlation tends to zero and the terms (b) can be neglected. The terms (c) produce "dips" in the integrand at

$$\tau-\sigma = -t_1, +t_2$$

but these occupy little of the total area of integration and may safely be neglected. Finally, then, we are left with

$$M = e^{c(\tau-\sigma) + c(\tau-\sigma+t_1-t_2) - 2c(0)} \qquad (73)$$

and again using (70), we find

$$M = \psi(\tau-\sigma)\, \psi(\tau-\sigma+t_1-t_2) \qquad (74)$$

This is exactly the second of the terms at (27), which gave the result at (34), or more briefly from (36),

(75) $$\overline{|A|^2} = (TW)^{-1} G(T, W)$$

This is the familiar noisy pedestal, but it is found only outside the central strip previously discussed.

The complete ambiguity diagram for random phase modulation, based on the results (72) and (75) is shown in Figure 10 on page 20.

It might at first appear that the above analysis is independent of the standard deviation of the actual phase angle and the bandwidth of its temporal fluctuation, as the results seem only to involve the total bandwidth of the signal as a whole. This is, in reality, untrue. The methods of approximation will be found, upon careful analysis, to assume a "reasonable" value for $c(0)$, i.e. an r.m.s. phase excursion just large enough to de-correlate the signal for large time-delays. This would imply phase-excursions of the order of one cycle, i.e. little more than jitter. Comparison of (65) and (70) and (68) show

(76) $$b = 2\pi^2 B^2 c(0)$$

and it will be appreciated that b is proportional to the square of the overall noise bandwidth. The same value of b could be achieved by means of a large mean squared phase excursion $c(0)$ and a correspondingly smaller rate of fluctuation B. Under these conditions, it is found that the approximation (i) extends over a wider central strip than before, as the interface between the two regions is really determined by B rather then $\sqrt{b}$. Thus a wider central strip would be cleared, but range resolution would be unnecessarily worsened. There seems little point in doing this, as it is wasteful of radar bandwidth.

To summarize, we have found that, assuming minimum phase excursion necessary to de-correlate the radar modulation at large time-shifts, the functional formula for ambiguity is given by

(77) $$Y(T,W) = \begin{cases} G(W^{-1}, T^{-1}), & |t| < W^{-1} \\ (TW)^{-1} G(T,W), & |t| > W^{-1} \end{cases}$$

approximately. To a graphical approximation, which would be exact for our conventional drawings, this could as well be written in the form

(78) $$Y(T,W) = G(W^{-1}, T^{-1}) + (TW)^{-1}\left[G(T,W) - G(W^{-1}, W)\right]$$

Comparison with (36) and (52) show the remarkable result

(79) $$2Z = X + Y$$

The ambiguity function for complex noise is the sum of the functions for amplitude noise and phase noise - strange bed-fellows in most mathematical problems.

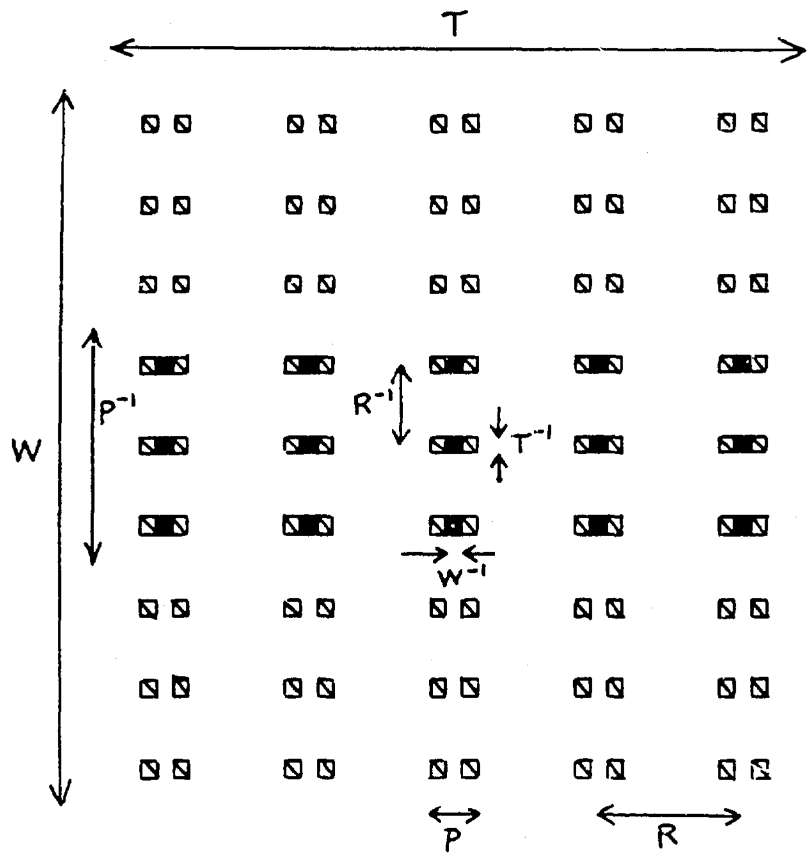


Fig 11. Ambiguity of coherent train of identical phase modulated bursts

T = total duration of signal W = bandwidth
P = duration of each burst R = repetition period

Within a burst the phase modulation is random

$(PW)^{-1}$ 1

The result (77), illustrated at Figure 10, lends itself to generalization in exactly the same way as the "complex noise burst" given at (36) and Figure 5. Various types of pulse-train are obtainable as described in section 9 merely by reading Y in place of Z. Thus Figure 11 shows the diagram for a finite succession of identically phase-modulated bursts, Figure 12 shows bursts of independent phase modulation, and finally Figure 13 is the diagram for a conventional incoherent pulsed radar with no random phase modulation except in the sense that each pulse is triggered at a random carrier phase-angle.

12. Elimination of "range side-lobes"

It goes almost without saying that the roles of time and frequency may be interchanged with impunity. This turns all diagrams through a right angle, and Figure 10 thus suggests a means of clearing a strip along the t-axis free of random side-lobes. The signal would consist of a spectrum of oscillations phased at random but smooth in amplitude versus frequency. In other words, we might start with a conventional pulse and then disperse the pulse by means of a filter, flat in amplitude but with an elaborate finely-structured phase characteristic. The power spectrum of the signal would be unaltered, and hence the temporal auto-correlation function would remain as clean as it was for the original pulse.

13. Skew signals

The ambiguity diagrams discussed in this paper all possess rectangular symmetry, but the method of convolution described in section 5 can be powerfully employed to obtain diagrams for skew signals such as linear frequency modulation. One could take as a primitive signal, for example, the linear frequency modulation

$$u(t) = e^{\pi i r t^2} \tag{80}$$

for which the instantaneous frequency is

$$f_i = \frac{1}{2\pi}\frac{d}{dt}(\pi r t^2) = rt \tag{81}$$

Substitution into (1a) yields

$$A = \delta(f - rt) \tag{82}$$

which is an oblique line of ambiguity. Convolution with this signal will shear any ambiguity diagram to any desired extent.

14. A Futility Theorem

There is continued speculation on the subject of ambiguity clearance. Like slums, ambiguity has a way of appearing in one place as fast as it is made to disappear in another. That it must be conserved is completely accepted, but the thought remains that ambiguity might be segregated in some unwanted part of the t-f plane where it will cease to be a practical embarrassment. We now put forward a line of reasoning, suggested by a similar approach of Robert Price, which should at least dispose of any grandiose clearance schemes.

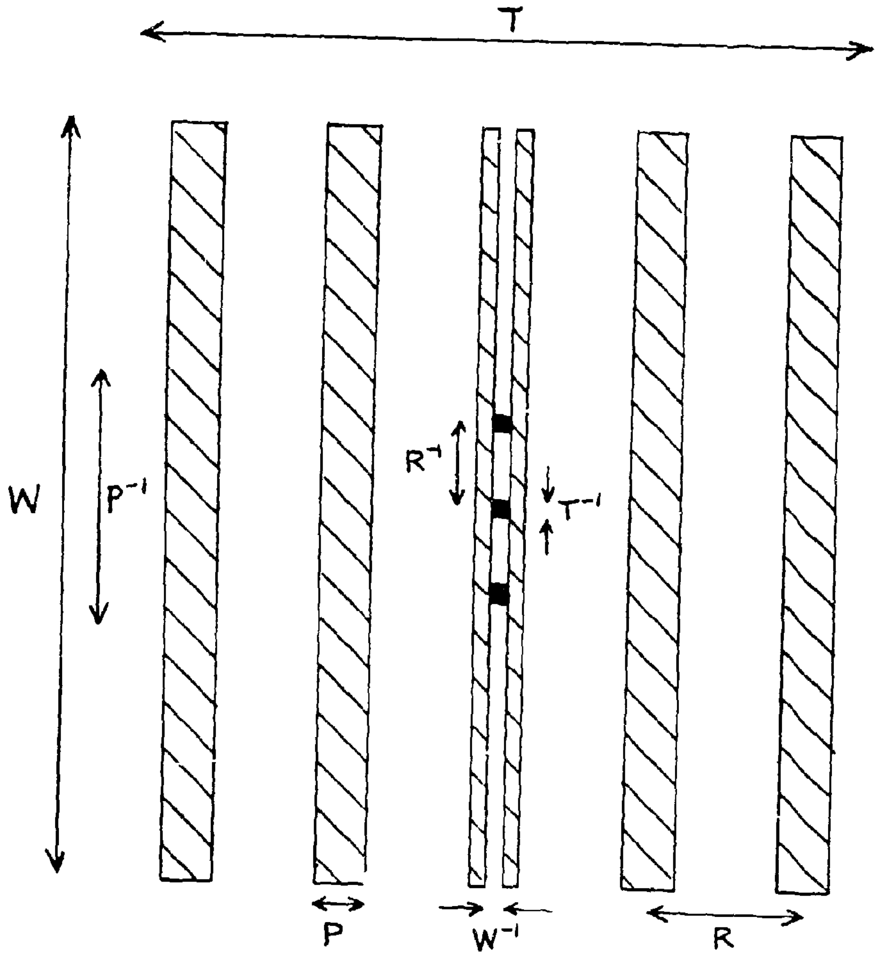
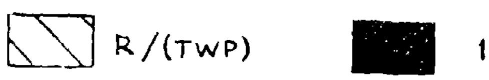


Fig 12. Ambiguity of train of independent bursts of random phase modulation

T = total duration of signal   W = bandwidth
P = duration of each burst   R = repetition period

Note the clear central strip



The reasoning is based on Siebert's Theorem, which we have not yet applied in any way. Since an ambiguity diagram is its own double Fourier Transform, we can apply a double convolution theorem, using for example a Gaussian weighting function. Thus, if H denotes any ambiguity function, Siebert's Theorem states

(83) $$H \leftrightarrow \tilde{H}$$

where twiddle denotes transposition of axes and arrow denotes Fourier Transformation with respect to t and f. From the property of Gaussian functions, we have

(84) $$G(R,L) \leftrightarrow G(R^{-1}, L^{-1})$$

and hence by the convolution theorem applied in both dimensions, we have

(85) $$H * G(R,L) \leftrightarrow \tilde{H} . \tilde{G}(L^{-1}, R^{-1})$$

As an illustration, consider the ambiguity function shown at Figure 14. The dotted rectangle represents

(86) $$G(R, R^{-1})$$

The part of the diagram inside the dotted rectangle is

(87) $$H . G(R, R^{-1}) = G(W^{-1}, T^{-1})$$

Transposing, transforming, and comparing with (85), we find

(88) $$H * G(R, R^{-1}) = G(T, W).$$

Inspection of the figure immediately verifies the truth of this equation, even though we perform the verification in terms of rectangles rather than ellipses. Wherever the dotted rectangle is moved, it will enclose the same amount of ambiguity until we reach the edges of the diagram.

As a reductio ad absurdum, suppose that a much larger area had been cleared of ambiguity. (It would appear at first that we already have a factor of almost four in hand, but we must remember that the diagrams are not truly black and white.) Then it would be possible to argue as follows.

Let $G(kR, kR^{-1})$ be a rectangle, <u>much</u> larger than $G(R, R^{-1})$, but which still encloses only the central pip of the diagram. Then

(89) $$H . G(kR, kR^{-1}) = G(W^{-1}, T^{-1})$$

and as previously,

(90) $$H * G(k^{-1}R, k^{-1}R^{-1}) = G(T, W)$$

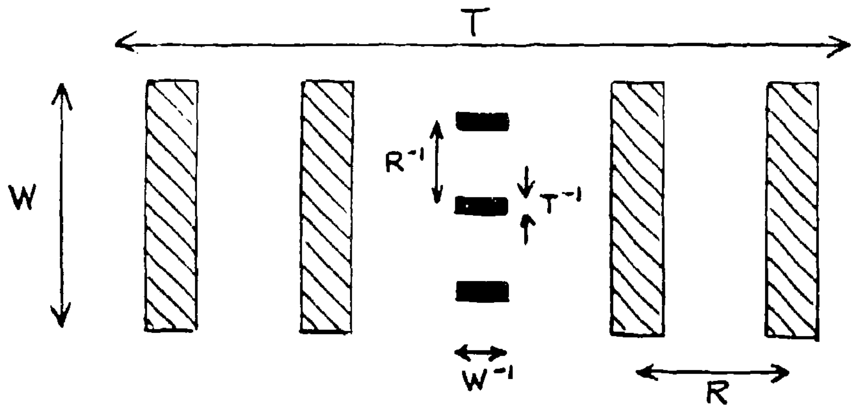


Fig 13. Ambiguity of incoherent pulse-train

T = duration of train   R = repetition period
W = bandwidth

(Degenerate form of Fig 12 when PW < 1)

R/T   1

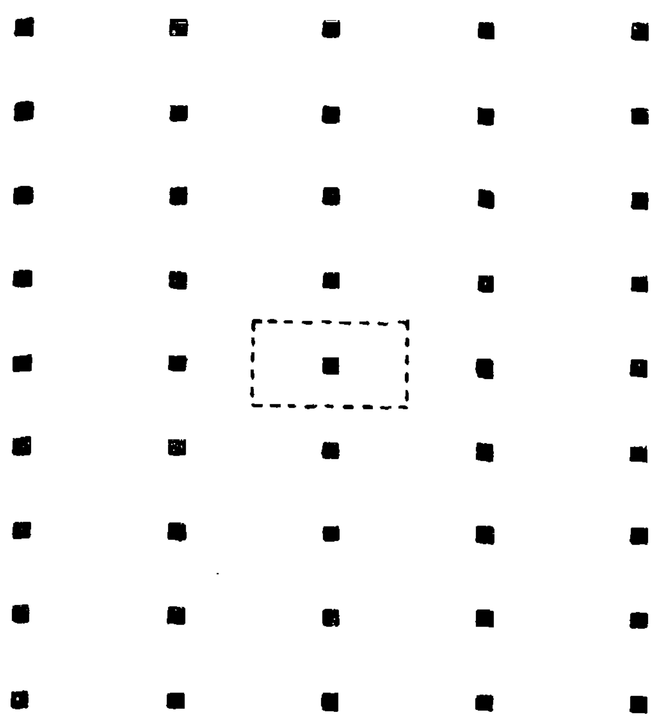


Figure 14

From the result on the right-hand side, we see that the Gaussian function on the left can never (save at the edges of the diagram) fit inside a clear area of H. But since, in area covered,

$$(91) \qquad \text{Area of } G(k^{-1}R, k^{-1}R^{-1}) \ll \text{Area of } G(kR, kR^{-1}),$$

this is contrary to hypothesis.

It follows from this line of reasoning that any ambiguity diagram with a small central pip, i.e. showing high resolution, must extend over a reciprocal area in the t-f plane, and the clearer the surround to the central zone, the more uniformly must be ambiguity be spread over the entire plane. No magic answers are to be found.

15. Acknowledgements

The author expresses gratitude to G. R. Whitfield for collaboration and stimulus in reaching all the results intuitively before any mathematical proofs were devised. Colleagues P. R. Wetherall and S. N. Higgins have also assisted, Mr Higgins' having devised much of Appendix 2.

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Appendix 1 - Moment generation for correlated normal variates

Let $x_1\ x_2\ \ldots\ x_s$ be s Gaussian random numbers with zero means, and not necessarily independent. Further assume that all co-variances

$$b_{k\ell} = \overline{x_k x_\ell} \tag{A-1}$$

are given. Then the joint distribution of the s variables is completely determined. However, it is often unnecessary to know the distribution explicitly. The moment-generating function (see ref. 2) is

$$g(y_1 \ldots y_s) = \int \ldots \int p(x_1 \ldots x_s)\, e^{i \sum x_k y_k}\, dx_1 \ldots dx_s$$

$$= \exp\left[-\tfrac{1}{2} \sum_k \sum_\ell b_{k\ell}\, y_k y_\ell\right] \tag{A-2}$$

where $p(x_1\ \ldots\ x_s)$ is the Gaussian probability density function. To obtain any desired moments of the x's in p, we merely differentiate g with respect to the corresponding y's and then put all the y's equal to zero. Hence, for example, with s = 4 we find

$$i^4\, \overline{x_1 x_2 x_3 x_4} = \frac{d^4}{dy_1 dy_2 dy_3 dy_4} \exp\left[-\tfrac{1}{2} \sum_k \sum_\ell b_{k\ell}\, y_k y_\ell\right]_{y_1 = y_2 = y_3 = y_4 = 0}$$

and after performing the necessary manipulation,

$$\overline{x_1 x_2 x_3 x_4} = \overline{x_1 x_2} \cdot \overline{x_3 x_4} + \overline{x_1 x_3} \cdot \overline{x_2 x_4} + \overline{x_1 x_4} \cdot \overline{x_2 x_3} \tag{A-3}$$

The application required in section 10 of the text is for

$$\begin{aligned} x_1 &= n(\tau) \\ x_2 &= n(\tau + t_1) \\ x_3 &= n(\sigma) \\ x_4 &= n(\sigma + t_2) \end{aligned}$$

where n is a stationary Gaussian random function. Hence

$$\begin{aligned} \overline{x_1 x_2 x_3 x_4} = {} & \psi(t_1)\psi(t_2) + \psi(\tau-\sigma)\psi(\tau-\sigma+t_1-t_2) \\ & + \psi(t_1+\tau-\sigma)\psi(t_2+\sigma-\tau) \end{aligned} \tag{A-4}$$

as shown at (27) and (48).

Appendix 2 - Complex noise

Little seems to have been published on the subject of complex random functions; most treatises on noise strangely fail to take full advantage of the complex notation which is commonplace in signal analysis. We must therefore define with some care what is meant by a complex normal variate and its generalization, the complex Gaussian random function.

Let x and y be normal variates of zero mean. Then

$$z = x + iy$$

will be called a complex normal variate if and only if

$$\overline{z^2} = 0. \tag{A-5}$$

This implies that x and y are uncorrelated but have the same standard deviation. The variance of z is defined to be

$$\overline{zz^*} = \overline{x^2} + \overline{y^2} = 2\,\overline{x^2} = 2\,\overline{y^2} \tag{A-6}$$

If z_1 and z_2 are two independent complex variates, we have

$$\overline{z_1 z_2} = 0 \tag{A-7}$$

because all four terms in the expansion (x_1x_2, y_1y_2, x_1y_2 and x_2y_1) have zero means. However, we shall show below that the above condition is true, not merely for independent complex variates, but for any pair of variates, whether independent or not.

The general multi-variate situation is set up by making a complex linear transformation of a set of independent complex numbers $z_1 \ldots z_s$. Let this be

$$n_k = \sum_i a_{ki} z_i \tag{A-8}$$

where the a's are complex constants. Then the n's are correlated complex numbers, and

$$n_k n_\ell = \sum_i a_{ki} z_i \sum_i a_{\ell i} z_i$$

Upon averaging, each term has a factor like (A-5) or (A-7) whether $k = \ell$ or not. Hence

$$\overline{n_k n_\ell} = 0 \tag{A-9}$$

invariably, and we may take this as characteristic of the complex multivariate distribution. If it holds good for any initial set of variables, it must continue to hold good after these have undergone any number of linear transformations.

Correlation between a pair of complex numbers must be expressed by the complex co-variance

$$\overline{n_1 n_2^*} \tag{A-10}$$

and in practical applications it is always the real part which is of prime importance. This is due to the fact that

$$\begin{aligned} |n_1 - n_2|^2 &= (n_1 - n_2)(n_1^* - n_2^*) \\ &= |n_1|^2 + |n_2|^2 - 2\,\mathcal{R}\,(n_1 n_2^*). \end{aligned} \tag{A-11}$$

Thus, for instance, when two waveforms are cross-correlated to obtain the best position of fit or minimum squared modulus of difference, it is

$$\mathcal{R} \int u_1 u_2^* \, dt$$

which must be maximized, and the imaginary part is not relevant. In many practical cases, the complex co-variance proves to be purely real. For example, it is readily shown that independent complex normal variates subjected to a purely real linear transformation yield numbers which have real co-variances. For let the a's in (A-8) be real. Then we have

$$n_k n_\ell^* = \sum_i a_{ki} z_i \sum_i a_{\ell i} z_i^*$$

and each term in the averaged product is either real or zero. It is interesting to note that, when

$$\overline{n_1 n_2^*}$$

is real, we have (denoting the real and imaginary parts by u and v) by hypothesis

$$\overline{u_1 v_2} = \overline{u_2 v_1} \tag{A-12}$$

but we also have

$$\overline{u_1 v_2} + \overline{u_2 v_1} = 0 \tag{A-13}$$

from (A-9). Thus, when the co-variance is real, we find that

$$\overline{u_1 v_2} = \overline{u_2 v_1} = 0 \tag{A-14}$$

It follows from the above discussion that stationary white complex noise, passed through a real linear circuit (stereo?), emerges with a purely real auto-correlation function. It is thus sufficient that the power spectrum of the noise should be symmetrical about zero frequency,

and we may then safely assume, from (A-14) or indeed from the physical insight provided by the thought of a stereo channel for quadrature components, that the real and imaginary parts of the noise are uncorrelated from each other at whatever instants of time we care to choose.

At (26) in the text, it is required to evaluate a fourth order complex moment of the form

$$\overline{n_1^* n_2 n_3 n_4^*}$$

By expanding into real and imaginary parts, assuming n to have a real auto-correlation function, and by using the results (A-3) and (A-14), it can be shown that

$$\overline{n_1^* n_2 n_3 n_4^*} = \psi_{12}\psi_{34} + \psi_{13}\psi_{24} \qquad \text{(A-15)}$$

where

$$\psi_{k\ell} = \overline{n_k n_\ell^*} \qquad \text{(A-16)}$$

This is how (27) was obtained. Whether or not equation (A-15) extends to noise whose co-variance is not real has not been investigated. It was not required in the paper.

PMW/NL
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